Quantified Beurling's uncertainty principle for Fourier transforms

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1 Introduction: Beurling's uncertainty principle

The uncertainty principle in harmonic analysis essentially states that:

A non-zero function and its Fourier transform cannot simultaneously be too sharply localized.

There are numerous precise mathematical formulations of this statement, see for instance [F-S, H-J].

Beurling's own version of the uncertainty principle is the following elegant theorem:

Theorem 1.1 (Beurling) Let $f \in L^1(\mathbb{R})$. If

$$\iint_{\mathbb{R}^2} |f(x)\hat{f}(\xi)| e^{|x\cdot\xi|} \, dx \, d\xi < \infty, \qquad (1.1)$$

then necessarily $f = 0$. //

Theorem 1.1 appears in Beurling's collected works [B, p.372] without a proof and the editors stated there that no proof was preserved. However, Hörmander later published Beurling's original proof in [Hö], based on notes he kept when Beurling explained it to him in a private conversation at some time during 1964-1968.

Beurling's Theorem 1.1. includes as particular instances other well-known uncertainty principles for Fourier transforms. Before giving some examples, let us point out that we fix the constants in the Fourier transform as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x\rangle} dx$$

Example 1.2. (Hardy's uncertainty principle, 1933)

Let $a \cdot b > 1$. Then

$$|f(x)| \lesssim e^{-a\frac{x^2}{2}} \text{ and } |\hat{f}(\xi)| \lesssim e^{-b\frac{\xi^2}{2}} \Rightarrow f = 0.$$

Indeed, let $\eta = a - \varepsilon > 0$, using $|x \cdot \xi| \le \frac{\eta}{2}|x|^2 + \frac{1}{2\eta}|\xi|^2$,

$$\iint_{\mathbb{R}^2} |f(x)||\hat{f}(\xi)| e^{|x\cdot\xi|} dx\, d\xi \lesssim \iint_{\mathbb{R}^2} e^{-\frac{\varepsilon}{2}|x|^2 - \frac{(b - \frac{1}{a-\varepsilon})|\xi|^2}{2}} dx\, d\xi$$

$$< \infty,$$

if we choose $\varepsilon > 0$ so small that $b \cdot a > 1 + \varepsilon \cdot b$. ///

Similarly,

Example 1.3 (Gelfand-Shilov type uncertainty principle)

Let Ω be a convex non-decreasing non-negative unbounded function on $[0, \infty)$. Its Young conjugate (or Legendre transform) is:

$$\Omega^*(\xi) = \sup_{x \ge 0} (\xi \cdot x - \Omega(x)).$$

(Then $\Omega(x) + \Omega^*(\xi) \ge x \cdot \xi$). Let $a \cdot b > 1$. Then

$$|f(x)| \lesssim e^{-\Omega(ax)} \text{ and } |\hat{f}(\xi)| \lesssim e^{-\Omega^*(bx)} \Rightarrow f = 0 \;///$$

$\boxed{2}$ Quantified Beurling's uncertainty principle

The aim of this talk is to discuss a generalization of Theorem 1.1. We start with some preliminaries. We denote as $\{h_\alpha\}_{\alpha\in\mathbb{N}^n}$ the Hermite orthonormal basis of $L^2(\mathbb{R}^n)$,

$$h_\alpha(x) = (-1)^{|\alpha|}\pi^{-\frac{n}{4}}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{|x|^2}{2}}\frac{\partial^\alpha}{\partial x^\alpha}(e^{-|x|^2}),\ \alpha\in\mathbb{N}^n.$$

If A is a real (symmetric) positive definite matrix, and $A^{1/2}$ its positive definite square root, we write

$$h_{A,\alpha}(x) = (\det A)^{1/4}\, h_\alpha(A^{1/2}x),$$

the A-skewed Hermite functions.

Theorem 2.1 (Neyt, V., 2026). Let $\omega:[0,\infty)\to[0,\infty)$ be a non-decreasing function such that $\varphi(t):=\omega(e^t)$ is convex and

$$\int_t^\infty \frac{\omega(s)}{s^2}\,ds \lesssim \frac{\omega(t)}{t},\qquad t\to\infty. \tag{2.1}$$

Suppose that $f\in L^1(\mathbb{R}^n)$ is such that

$$\iint_{\mathbb{R}^{2n}} |f(x)||\hat{f}(\xi)|\,\frac{e^{|\langle x,\xi\rangle|}}{e^{\omega(|x|+|\xi|)}}\,dx\,d\xi<\infty. \tag{2.2}$$

then f must necessarily be of the form

$$f(x) \doteq e^{-\frac{\langle Ax, x\rangle}{2}} \quad P(x) = \sum_{\alpha \in \mathbb{N}^n} H_A(f,\alpha)\, h_{A,\alpha}(x) \quad (2.3)$$

where A is a positive definite real matrix, P is an entire function satisfying the bound

$$|P(z)| \lesssim e^{c\,\omega(|z|)} \qquad z \in \mathbb{C}^n, \qquad (2.4)$$

and the A-skewed Hermite coefficients of f are such that

$$|H_A(f,\alpha)| \lesssim \sqrt{\alpha!}\, e^{-r\varphi^*\left(\frac{|\alpha|}{r}\right)}, \quad \alpha \in \mathbb{N}^n, \quad (2.5)$$

with φ^* the Young conjugate of φ.

Here c and r are effective constant (explicit values can be given in terms of ω).

Note that by taking $\omega \equiv 0$, we obtain $\varphi^* \equiv \infty$, so that one recovers Beurling's. The case $\omega(t) = \lambda \log(1+t)$ delivers the following quantified uncertainty principle due to Bonami, Demange, and Jaming.

Corollary 2.2 ([B-D-J]) Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then

$$\iint_{\mathbb{R}^{2n}} \frac{|f(x)|\,|\hat{f}(\xi)|}{(1+|x|+|\xi|)^\lambda} e^{|\langle x,\xi\rangle|}\,dx\,d\xi < \infty$$

if and only if $f(z) = e^{-\frac{\langle Az,z\rangle}{2}} P(z)$ where A is a positive definite symmetric matrix and P is a polynomial of degree $< (\lambda - n)/2$. //

In the remainder of this talk we discuss some ideas connected with the proof of Theorem 2.6. We assume from now on that ω satisfies the assumptions from Theorem 1.2. For the sake of simplicity, we only consider the one dimensional case $n=1$.

3 Some tools related to ω

We discuss in this section some properties of the weight function ω that will play a crucial role in our proof of Theorem 2.6.

The relevance of condition (2.1) lies in the fact that it yields the validity of the following version of the Phragmén-Lindelöf principle on sectors, essentially shown in [Theorem 3.1, N-T-V].

Given any $\theta>0$, we consider the sector

$$S_\theta = \{ z : -\frac{\theta}{2} < \arg z < \frac{\theta}{2} \}.$$

Lemma 3.1 (Neyt, Toft, V., 2025) Let $0<\theta\leq\pi$. There are constants $L=L(\omega,\theta)\geq 1$ and $\Lambda=\Lambda(\omega,\theta)>0$ such that for any continuous function $F\in C(\overline{S}_\theta)$ that is holomorphic on S_θ, the bounds $(\lambda>0)$

$$|F(z)|\leq M\cdot e^{\lambda\omega(|z|)}, \quad z\in\partial S_\theta, \qquad (3.1)$$

and, for each $\varepsilon>0$,

$$\liminf_{r\to\infty} e^{-\varepsilon r^{\frac{\pi}{\theta}}} \sup_{\substack{|z|=r \\ z\in S_\theta}} |F(z)| = 0 \qquad (3.2)$$

imply that

$$|F(z)|\leq \Lambda\cdot M e^{L\lambda\omega(|z|)}, \quad z\in S_\theta \quad (3.3).$$

Remark 3.2 If $\theta\in(0,\pi/2]$, which is the case we actually use, Lemma 3.1 holds under the milder condition

$$\int_t^\infty \frac{\omega(s)ds}{s^3} \lesssim \frac{\omega(t)}{t^2} \qquad (3.4)$$

In fact (3.4) is optimal in the sense that if each function $F\in C(\overline{S}_\theta)$ analytic in S_θ, $0<\theta\leq\frac{\pi}{2}$ one has (3.1) and (3.2) implies (3.3), then (3.4) must hold true. ////

The second important result we employ is the following quantified version of Hardy's uncertainty principle, obtained in [Theorem 1.3, N-T-V].

Lemma 3.3 (Neyt, Toft, V., 2025) A function $f \in L^2(\mathbb{R})$ satisfies

$$|f(x)| \lesssim e^{-\frac{x^2}{2} + \lambda\omega(|x|)} \text{ and } |\hat{f}(\xi)| \lesssim e^{-\frac{\xi^2}{2} + \lambda\omega(|\xi|)}$$

for some $\lambda > 0$ if and only if its Hermite coefficients satisfy

$$H(f, j) \lesssim \sqrt{j!}\, e^{-\frac{1}{r}\varphi^*(rj)} \quad (3.5)$$

for some $r > 0$. ////

Remark 3.4 Once again the condition (2.1) on ω can be weakened to (3.4) in Lemma 3.3. ////

4 Comments on the proof of Theorem 2.1

In this last part of the talk we comment on our three main steps in our proof of Theorem 2.1.

The first observation is that (2.2) readily yields that both f and $\hat{f}$ are entire functions. Secondly, by regularizing f convolving with a Gaussian one may assume without lossing (much) generality that f is an entire function of order 2 (this requires technical justification but I ask the reader to accept it in order to explain the main underlying ideas). It is also easy to see that $f \in L^2(\mathbb{R})$. Under this extra assumptions, one deduces using properties of ω and the Phragmén-Lindelöf principle Lemma 3.1 that

Lemma 4.1 The entire function

$$F(z) = f(z) \cdot f(iz) \tag{4.1}$$

satisfies the bound

$$|F(z)| \lesssim e^{C\omega(|z|)} \tag{4.2}$$

for some absolute constant $C = C_\omega > 0$. ///

Entire functions satisfying bounds (4.2) are usually called (ω)-ultrapolynomials and are symbols of so-called ultradifferential operators (see e.g. [P-P-V]).

One can use (2.1) to show that F has genus 0, so that it has a product representation over its zeros. Studying its zero distribution and using that f has order 2, (4.1), (4.2), and (2.1) can be used to show that

Lemma 4.2 f must be of the form
$$f(z) = e^{-\frac{az^2}{2}} P(z), \quad (4.3)$$
where P is an (ω)-ultrapolynomial, that is, it satisfies the bound
$$|P(z)| \lesssim e^{c\omega(|z|)} \quad (4.4)$$
for some $c>0$. Here $\operatorname{Re} a > 0$. ///

To complete the proof of Theorem 2.1, one has to show that $\operatorname{Im} a = 0$ and that the a-skewed Hermite coefficients of f satisfy the required decay bounds (2.5). For it, we need to work the the Gelfand-Shilov space $S_{\frac{1}{2}} := S^{\frac{1}{2}}_{\frac{1}{2}}$. The simplest description of $S_{\frac{1}{2}}$ is via Fourier transform [C-C-K]
$h \in S_{\frac{1}{2}}$ if and only if there is some $b>0$ s.t. both $h(x) = O(e^{-b|x|^2})$ and $\hat{h}(\xi) = O(e^{-b|\xi|^2})$.
We refer to [G-S, P-P-V] for properties of $S_{\frac{1}{2}}$. One first work with a change of variables so that
$$h(\xi) = f\left(\frac{z}{\sqrt{2a}}\right) = e^{-\frac{\xi^2}{4}} H(\xi),$$

where the entire function H satisfies the same type of bound as P. We can interpret H as a multiplier operator on the ultradistribution space $S'_{1/2}$. Using the bound $H(\xi) = O(e^{c'\omega(|\xi|)})$, one can estimate the coefficients of H in its power series expansion and see that

$$H(iD) : S'_{1/2} \longrightarrow S'_{1/2}$$

is a well-defined continuous linear operator given as an infinite order differential operator (ultra-differential operators). This allows us to view

$$\hat{h}(u) = 2\sqrt{\pi}\, H(iD)\left(e^{-u^2}\right)$$
$$= e^{-\frac{u^2}{2}} \sum_{j=1}^{\infty} c_j h_j(u),$$

where the coefficients c_j satisfy bounds of the same quality as (3.6). Lemma 3.3 and change variables back lead to (3.5), but with a still complex. However, substituting in (2.2) and using $\omega(t) = \Theta(t)$ which follows from (2.1) yield $a \in \mathbb{R}_+$!

A second application of Lemma 3.3 completes the proof.

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